Exam – Mathematics II – Version A – 06/12/2024

Sketch

Part I

1 a) 5 and 0 ; 1 and 2

b) $D_f = \{(x,y) \in \mathbb{R}^2 : \underbrace{x \neq 0}_{x^2 > 0} \wedge \underbrace{x \neq \pm 1}_{\ln x^2 \neq 0}\}$

c) $Q(x,y) = 25x^2 - 30xy + 9y^2$

$$\begin{pmatrix} 25 & -15 \\ -15 & 9 \end{pmatrix}$$

- Q is semi-positively defined
- Q degree 2 ; $2\,Q(x,y)$

d) $(2, \frac{1}{2})$ for instance

- $\text{int}\ \Omega = \{(x,y) \in \mathbb{R}^2 : x \in \left]1,2\right[\wedge \frac{1}{x} < y\}$
- Compact.

e) $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Weierstrass; compact.

f) $-\frac{1}{2}$; 0 ; is not continuous at $(0,0)$

g) $\left(1+\frac{1}{2n}\right)^n \xrightarrow{n} e^{1/2}$

$\frac{1-n\pi}{2n} \xrightarrow{n} -\frac{\pi}{2}$

$\lim_{n\to+\infty} f(u_n) = -2 \cdot e^{1/2} + \cos(-\pi/2)$

$= -2\sqrt{e}$

h) $\frac{\partial f}{\partial x}(1,2) = D_{(1,0)} f(1,2) = \frac{1}{3}$

$\frac{\partial f}{\partial y}(1,2) = D_{(0,1)} f(1,2) = 0.$

i) $f(x,y) = \cos(x^2+y) + y^3$

j) $D_2 f(0,0)(h_1,h_2) = 0 h_1^2 + 0 h_1 h_2 + 9 h_2^2$

k) $\frac{df}{dt}(t) = 2t + 5t^4$

l) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ global

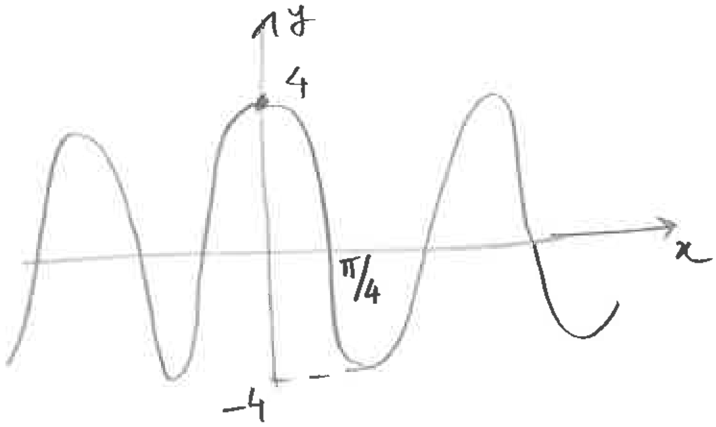
m) $\int_1^2 \int_1^{x^2} e^{x+y}\, dy\, dx = \int_1^4 \int_{\sqrt{y}}^2 e^{x+y}\, dx\, dy$

n) Malthus law

$p(t) = 20\, e^{3t}, \; t \in \mathbb{R}_0^+$

o) increasing

p) $y(x) = 4 \cos 2x$



Part II

(Sylvester theorem)

a) $\Delta_1 = 1 > 0$

$\Delta_2 = \alpha$

$\Delta_3 = 1 . \det \begin{pmatrix} \alpha & 4 \\ 4 & \alpha \end{pmatrix} = \alpha^2 - 16$

α	$-\infty$	-4		0		4	$+\infty$
Δ_1	+		+	+	+		+
Δ_2	−		−	0	+		+
Δ_3	+		−	−	−		+
Q	UND		UND	UND	UND		P.D

Using the leading minors method, we can conclude that

- $\alpha \in]-\infty, 4[\setminus \{-4\} \implies$ Q is undefined
- $\alpha \in]4, +\infty[\implies$ Q is Posit defined.

1b) $A\begin{pmatrix}0\\1\\1\end{pmatrix} = 3\begin{pmatrix}0\\1\\1\end{pmatrix}$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & 4\\ 0 & 4 & \alpha\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{pmatrix}0\\3\\3\end{pmatrix} \Leftrightarrow \begin{cases}0=0\\ \alpha+4=3\\ 4+\alpha=3\end{cases} \Leftrightarrow \boxed{\alpha=-1}$$

2a) $\boxed{y>x}$ ← open set. f is continuous because $f(x,y)$ is given by the ratio of continuous maps whose denominator is diff. from zero.

$\boxed{y<x}$ f is constant $\Rightarrow$ f is continuous

open set

$\boxed{x=y}$. Let $a\in\mathbb{R}$

$a\neq 0$ $\lim\limits_{(x,y)\to(a,a)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} 0 = 0$

$a=0$: Squeezing theorem:

$$0 \leq \left|\frac{x^2\,(x-y)}{\sqrt{x^2+y^2}}\right| \leq \frac{(x^2+y^2)\,|x-y|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\cdot|x-y|$$

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\cdot|x-y| = 0$$

then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$

$\therefore$ f is continuous in $\mathbb{R}^2$

2b)

$$y \le x \implies f(x,y) = 0$$

$$y > x \implies f(x,y) = \frac{x^2\,(x-y)}{\sqrt{x^2+y^2}} \le 0$$

$$\Downarrow$$

f has a global maximum (0).

3a) Critical points:

$$\nabla f(x,y) = \left(y - 2x \cdot \ln y \,;\; x - \frac{x^2}{y}\right) \qquad y>0$$

$$\nabla f(x,y) = \bar{0} \iff \begin{cases} y - 2x \ln y = 0 \\ x - \dfrac{x^2}{y} = 0 \end{cases} \iff \begin{cases} \text{———} \\ x\left(1 - \dfrac{x}{y}\right) = 0 \end{cases}$$

$$\iff \underbrace{\begin{cases} y = 0 \\ x = 0 \end{cases}}_{\text{impossible}} \vee \begin{cases} x\,(1 - 2\ln x) = 0 \\ y = x \end{cases} \iff \begin{cases} 2 \ln x = 1 \\ x = y \end{cases} \iff \begin{cases} x = \sqrt{e} \\ y = \sqrt{e} \end{cases}$$

Critical point. $(\sqrt{e}, \sqrt{e})$

Classification:

$$H_f(x,y) = \begin{pmatrix} -2 \ln y & 1 - \dfrac{2x}{y} \\ 1 - \dfrac{2x}{y} & + \dfrac{x^2}{y^2} \end{pmatrix}$$

$$H_f(\sqrt{e}, \sqrt{e}) = \begin{pmatrix} -2 \ln e^{1/2} & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$\Delta_1 = -1$

$\Delta_2 = -2 < 0$

$H_f(\sqrt{e}, \sqrt{e})$ is undefined

$\Downarrow$

$(\sqrt{e}, \sqrt{e})$ is a saddle-point

$\Downarrow$

f has no extreme in D

Lagrange multipliers method.

3b) $\mathcal{L}(x, y, \lambda) = xy - x^2 \ln y - \lambda(xy - 1)$

$\nabla \mathcal{L}(x, y, \lambda) = \overline{0} \Leftrightarrow \begin{cases} y - 2x \ln y - \lambda y = 0 \\ x - \frac{x^2}{y} - \lambda x = 0 \\ xy = 1 \end{cases}$

$\Leftrightarrow \begin{cases} \_\_\_ \\ x = 0 \\ \text{impossible} \end{cases} \vee \begin{cases} y - 2x \ln y - \left(1 - \frac{x}{y}\right) y = 0 \\ 1 - \frac{x}{y} = \lambda \\ \_\_\_ \end{cases}$

$\Leftrightarrow \begin{cases} \cancel{y} - 2x \ln y - \cancel{y} + x = 0 \\ 1 - \frac{x}{y} = \lambda \\ \_\_\_ \end{cases} \Rightarrow \begin{cases} x = 0 \\ \_\_\_ \end{cases} \vee \begin{cases} -2 \ln y + 1 = 0 \\ \_\_\_ \end{cases}$

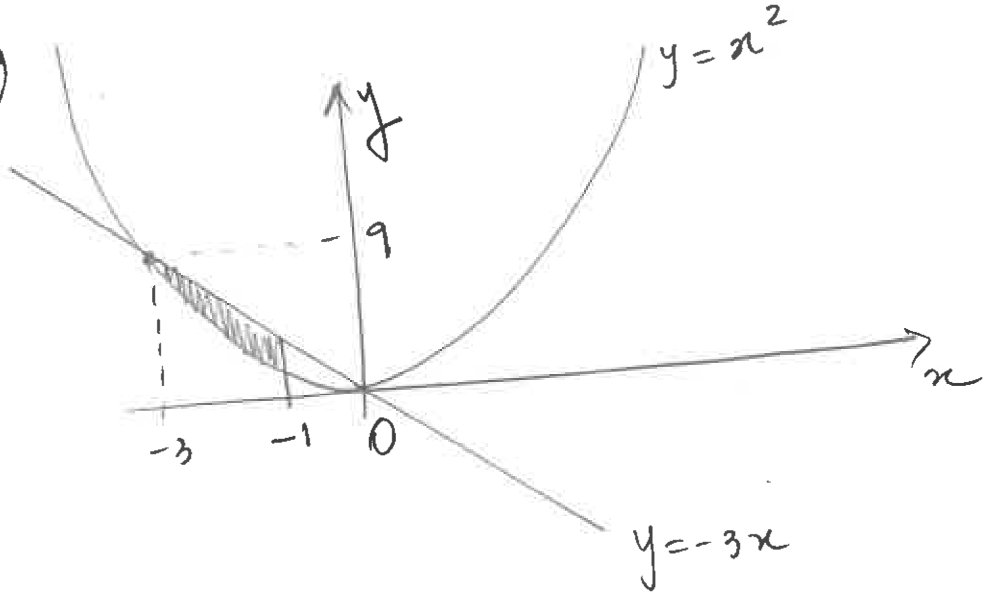
impossible

$\Leftrightarrow \begin{cases} 2 \ln y = 1 \\ \_\_\_ \end{cases} \Leftrightarrow \begin{cases} y = \sqrt{e} \\ x = \frac{1}{\sqrt{e}} \end{cases} \qquad \lambda = 1 - \frac{1}{\sqrt{e} \cdot \sqrt{e}} = 1 - \frac{1}{e}$

$$f\left(\frac{1}{\sqrt{e}}, \sqrt{e}\right) = \frac{1}{\sqrt{e}} \cdot \sqrt{e} - \frac{1}{e} \cdot \ln \sqrt{e}$$

$$= 1 - \frac{1}{e} \cdot \frac{1}{2} = 1 - \frac{1}{2e}$$

4a)



4b) Type I

$$\int_{-3}^{-1} \int_{x^2}^{-3x} \frac{e^{y/x}}{x} \, dy \, dx = \int_{-3}^{-1} \left[e^{y/x} \right]_{y=x^2}^{y=-3x} dx$$

$$= \int_{-3}^{1} e^{-3} - e^{x} \, dx = \left[e^{-3} x - e^{x} \right]_{-3}^{-1}$$

$$= e^{-3}(-1) - e^{-1} - \left(e^{-3} \cdot (-3) - e^{-3}\right)$$

$$= -\cancel{e^{-3}} - e^{-1} + 3e^{-3} + \cancel{e^{-3}} = 3e^{-3} - \frac{1}{e}$$

5

$$\begin{cases} xy' + y = xe^x \\ y(1) = 1 \end{cases}$$

$$xy' + y = xe^x \underset{x\neq 0}{\Leftrightarrow} y' + \underbrace{\frac{1}{x}}_{a(x)} y = \underbrace{e^x}_{b(x)}$$

$$y(x) = \frac{\int e^{\int a(x)dx} \cdot b(x)\, dx + C}{e^{\int a(x) dx}} \Leftrightarrow$$

$$y(x) = \frac{\int e^{\int \frac{1}{x} dx} \cdot e^x\, dx + C}{e^{\int \frac{1}{x} dx}} \Leftrightarrow$$

$$y(x) = \frac{\int x \cdot e^x\, dx + C}{e^{\ln|x|}} \Leftrightarrow y(x) = \frac{e^x x - e^x + C}{x}$$

$$\Leftrightarrow y(x) = \frac{e^x (x-1)}{x} + \frac{C}{x}$$

$$y(1) = 1 \Leftrightarrow \frac{e^1 (1-1)}{1} + \frac{C}{1} = 1 \Leftrightarrow C = 1$$

$$\therefore y(x) = \frac{e^x (x-1)}{x} + \frac{1}{x}$$

$D = \underbrace{\mathbb{R}^+}$

maximal open interval containing $t = 1$.

<u>the end!</u>